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A Combined Spectral-Finite Element Method for Solving Two-Dimensional Unsteady Navier–Stokes Equations

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In this paper, a combined Fourier spectral-finite element method is proposed for solving two-dimensional, semi-periodic, unsteady Navier–Stokes equations. The convergence is proved and the numerical results are presented.

I. INTRODUCTION

When we study boundary layer stability, unsteady seperation of viscous fluid flow, flow past a suddenly heated vertical plate, and other fluid dynamics problems, we have to solve Navier–Stokes equations with semi-periodic boundary conditions. Many people have developed numerical methods to solve these problems. For instance, Murdock [1, 2], Ingham [3, 4], Beringen [5], Milinazo and Saffman [6], and Ben-yu Guo [7] used combined spectral-finite difference methods. It means that spectral methods are adopted in the periodic directions, while finite difference methods are used in the non-periodic directions. Recently, Canuto, Maday, and Quarteroni [8, 9], Ben-yu Guo and Wei-ming Cao [10], and others, studied the combined spectral (or pseudospectral)-finite element methods. Another useful method is to use combined Fourier–Chebyshev approximations [11–13]. In this method, Fourier approximations are used in the periodic directions, while Chebyshev approximations are used in the other directions.

In this paper, we generalize the work of [8–10] to construct a combined Fourier spectral-finite element scheme for solving two-dimensional, semi-periodic, unsteady Navier–Stokes equations. Surely such problems can also be solved by spectral-finite difference methods or combined Fourier–Chebyshev approximations. But it is difficult to extend them to three-dimensional problems with non-rectangular domains. On the contrary, the method in this paper can be generalized easily to more complicated problems.

In Section II, we construct the combined spectral-finite element scheme. In Section III, we state the convergence theorem. The numerical results are presented in Section IV. They show that such a scheme gives much better performances than the usual finite element method does. We prove the convergence in the final two sections.

Some lemmas proved in this paper are also useful for other relevant problems.

II. THE SCHEME

Let

$$I=\{x/0<x<1\},\quad \tilde{I}=\{y/0<y<2\pi\},\quad \Omega=I\times\tilde{I},$$

and $U=(U^{(1)}, U^{(2)})$, P, f, and $\nu>0$ be the velocity, the ratio of pressure over density, the body force, and the kinetic viscosity, respectively. We consider the two-dimensional unsteady Navier–Stokes equations as follows:

$$\begin{aligned}&\frac{\partial U}{\partial t}+(U\cdot\nabla)U+\nabla P-\nu\nabla^2 U=f,\\ &\qquad \text{in}\quad \Omega\times(0,T],\\ &\nabla\cdot U=0,\qquad \text{in}\quad \Omega\times[0,T],\\ &U|_{t=0}=U_0,\qquad \text{in}\quad \Omega.\end{aligned}\tag{2.1}$$

Assume that U, P, and f have the period 2π for the variable y, and that

$$U(0,y,t)=U(1,y,t)=0,\qquad \forall (y,t)\in\tilde{I}\times[0,T].$$

In addition, P satisfies the following normalizing condition:

$$\iint_\Omega P(x,y,t)\,dx\,dy=0,\qquad \forall t\in[0,T].$$

For $\mu\geqslant 0$, we denote by $H^\mu(\Omega)$, $\|\cdot\|_\mu$ and $|\cdot|_\mu$ the classical

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Sobolev space, its norm, and semi-norm, respectively. In particular, we define $\mathscr{L}^2(\Omega)=H^0(\Omega)$ with the norm $\|\cdot\|$ and the inner product $(\cdot,\cdot)$. Furthermore, define

$$C_p^\infty(\Omega)=\{\eta\in C^\infty(\Omega)/\eta \text{ has the period } 2\pi \text{ for the variable } y\},$$

$$C_{0,p}^\infty(\Omega)=\{\eta\in C_p^\infty(\Omega)/\eta(0,y)=\eta(1,y)=0, \ \forall y\in\tilde{I}\}.$$

$H_p^\mu(\Omega)$ and $H_{0,p}^\mu(\Omega)$ denote the closures of $C_p^\infty(\Omega)$ and $C_{0,p}^\infty(\Omega)$ in $H^\mu(\Omega)$, respectively. We also define

$$\bar{\mathscr{L}}^2(\Omega)=\left\{\eta\in\mathscr{L}^2(\Omega)\Big/\iint_\Omega \eta\, d\Omega=0\right\}.$$

The generalized solution of (2.1) is the pair $(U(t),P(t))\in[H_{0,p}^1(\Omega)]^2\times\bar{\mathscr{L}}^2(\Omega)$ satisfying

$$\begin{aligned}
&\frac{d}{dt}(U(t),v)+((U(t)\cdot\nabla)U(t),v)\\
&\quad -b(v,P(t))+\nu a(U(t),v)\\
&\quad =(f(t),v), \qquad \forall v\in[H_{0,p}^1(\Omega)]^2,\\
&b(U(t),\omega)=0, \qquad \forall\omega\in\bar{\mathscr{L}}^2(\Omega),\\
&U(0)=U_0,
\end{aligned}\tag{2.2}$$

where

$$a(\eta,\xi)=\iint_\Omega(\nabla\eta)(\nabla\xi)\,d\Omega,$$

$$b(\eta,\xi)=\iint_\Omega(\nabla\cdot\eta)\,\xi\,d\Omega.$$

We introduce a trilinear form $J(\cdot,\cdot,\cdot)$: $[(H^1(\Omega))^2]^3\to R^1$ as follows:

$$J(\eta,\varphi,\xi)=\tfrac{1}{2}[((\varphi\cdot\nabla)\eta,\xi)-((\varphi\cdot\nabla)\xi,\eta)].$$

Clearly, we have

$$J(\eta,\varphi,\xi)+J(\xi,\varphi,\eta)=0,\tag{2.3}$$

and if $\nabla\cdot\varphi=0$, then

$$J(\eta,\varphi,\xi)=((\varphi\cdot\nabla)\eta,\xi).$$

Thus (2.2) is equivalent to

$$\begin{aligned}
&\frac{d}{dt}(U(t),v)+J(U(t),U(t),v)\\
&\quad -b(v,P(t))+\nu a(U(t),v)\\
&\quad =(f(t),v), \qquad \forall v\in[H_{0,p}^1(\Omega)]^2,\\
&b(U(t),\omega)=0, \qquad \forall\omega\in\bar{\mathscr{L}}^2(\Omega),\\
&U(0)=U_0.
\end{aligned}\tag{2.4}$$

For numerical solution of (2.4), we approximate the first equation directly. While to tackle the incompressible constraint (i.e., the second equation of (2.4)), we adopt the idea of artificial compression (see Téman [14]), that is, to approximate the incompressible condition by the equation

$$\beta\frac{d}{dt}(P(t),\omega)+b(U(t),\omega)=0, \qquad \forall\omega\in\bar{\mathscr{L}}^2(\Omega),$$

where $\beta>0$ is a small parameter.

Now we construct the scheme. First, we devide I into M_h subintervals, with the nodes $0=x_0<x_1<\cdots<x_{M_h}=1$. Let $I_l=(x_{l-1},x_l)$, $h_l=x_l-x_{l-1}$, $h=\max_{1\leqslant l\leqslant M_h}h_l$, and $h'=\min_{1\leqslant l\leqslant M_h}h_l$. We assume, furthermore, that there exists a positive constant d independent of the divisions of I, such that $h/h'\leqslant d$. For any integer $k\geqslant 0$, we denote by P_k the set of all the polynomials defined on R^1 of degree $\leqslant k$. We define the finite element subspaces in the non-periodic directions as

$$\tilde{S}_h^k=\{\eta/\eta|_{I_l}\in P_k,\ 1\leqslant l\leqslant M_h\},$$
$$S_h^k=\tilde{S}_h^k\cap H_0^1(I).$$

Suppose N is a positive integer; we define the subspaces for Fourier spectral approximations as follows:

$$S_N=\operatorname{Span}\{e^{ijy}/|j|\leqslant N\}.$$

Let $\alpha=(N,h,k)$, we define the following finite dimensional subspace as the trial function space for the velocity

$$X_\alpha=\{S_h^{k+1}\otimes S_N\}\times\{S_h^{k+2}\otimes S_N\}.$$

The trial function space for the pressure P is

$$Y_\alpha=\{S_h^k\otimes S_N\}\cap\bar{\mathscr{L}}^2(\Omega).$$

Let P_N be the orthogonal projection from $\mathscr{L}^2(\tilde{I})$ onto S_N.

Π_h^k is the piecewise Lagrange interpolation of order k from $C(\tilde{I})$ onto S_h^k; i.e., for any $\eta \in C(\tilde{I})$, $\Pi_h^k \eta \in S_h^k$ satisfies

$$(\Pi_h^k \eta)\left(x_{l-1} + \frac{m}{k} h_l\right) = \eta\left(x_{l-1} + \frac{m}{k} h_l\right), \quad 0 \leqslant m \leqslant k,\ 1 \leqslant l \leqslant M_h.$$

We obviously have that

$$P_N \circ \Pi_h^k = \Pi_h^k \circ P_N.$$

Let τ be the mesh size in time t and $S_\tau = \{t = l\tau / 0 \leqslant l \leqslant [T/\tau]\}$. Define

$$\eta_t(t) = \frac{1}{\tau}(\eta(t+\tau) - \eta(t)).$$

A fully discrete spectral-finite element scheme for (2.4) is to find the pair $(u(t), p(t)) \in X_\alpha \times Y_\alpha$ for all $t \in S_\tau$, such that

$$\begin{aligned}
&(u_t(t), v) + J(u(t) + \delta\tau u_t(t), u(t), v) \\
&\quad - b(v, p(t) + \theta\tau p_t(t)) \\
&\quad + \nu a(u(t) + \sigma\tau u_t(t), v) \\
&\quad = (P_N \circ \Pi_h^k f(t), v), \qquad \forall v \in X_\alpha, \qquad (2.5)\\
&\beta(p_t(t), \omega) + b(u(t) + \theta\tau u_t(t), \omega) \\
&\quad = 0, \qquad \forall \omega \in Y_\alpha, \\
&u(0) = P_N \circ \Pi_h^{k+1} U_0, \qquad p(0) = 0,
\end{aligned}$$

where $\delta, \sigma \geqslant 0$ and $\theta > \frac{1}{2}$ are parameters.

Remark 1. Let φ_α be the L^2-projection operator from $\mathscr{L}^2(\Omega)$ onto Y_α. Then it follows from the second equation of (2.5) that

$$p(t+\tau) = p(t) - \frac{\tau}{\beta}[\theta\varphi_\alpha(\nabla \cdot u(t+\tau)) + (1-\theta)\varphi_\alpha(\nabla \cdot u(t))]. \qquad (2.6)$$

By substituting the above formula into the first equation of (2.5), we obtain

$$\begin{aligned}
&(u(t+\tau), v) + \delta\tau J(u(t+\tau), u(t), v) \\
&\quad + \frac{\theta^2\tau^2}{\beta}(\varphi_\alpha(\nabla \cdot u(t+\tau)), \varphi_\alpha(\nabla \cdot v)) \\
&\quad + \sigma\nu\tau a(u(t+\tau), v) \\
&\quad = R(t)(v), \qquad \forall v \in X_\alpha, \qquad (2.7)
\end{aligned}$$

where $R(t)$ is a linear form defined on X_α and depends only on $u(t)$, $p(t)$, and $f(t)$. Clearly, if $R(t) = 0$, then by putting $v = u(t+\tau)$ in (2.7), we have from (2.3) that

$$\|u(t+\tau)\|^2 + \frac{\theta^2\tau^2}{\beta}\|\varphi_\alpha(\nabla \cdot u(t+\tau))\|^2 + \sigma\nu\tau\,|u(t+\tau)|_1^2 = 0.$$

Hence $u(t+\tau) = 0$. This implies that (2.7) has exactly one solution. As soon as $u(t+\tau)$ is found, we can obtain $p(t+\tau)$ by (2.6) immediately. In this way, we can solve the velocity and the pressure separately. This is, indeed, one of the advantages of the artificial compression treatment.

Remark 2. We observe from (2.7) that the artificial compression coefficient β is, in fact, a penalty parameter for restraint $\nabla \cdot U = 0$. Usually, a penalty method takes the term $1/\varepsilon(\nabla \cdot U, \nabla \cdot v)$ $(\varepsilon \to 0)$ to be the penalty function. But its numerical result is not good. It is interesting that if we calculate the integrations in the penalty function by a rough quadrature rule instead of calculating them exactly, then the results become better [15, 16]. This method is called the selective reduced integration and penalty method (or RIP method). The projection operator φ_α in the penalty function $(\theta^2\tau^2/\beta)(\varphi_\alpha(\nabla \cdot u(t+\tau)), \varphi_\alpha(\nabla \cdot v))$ in (2.7) is just an expression of the idea of the RIP method.

III. THE CONVERGENCE THEOREM

For describing the errors of the numerical solutions, we introduce the notation,

$$\begin{aligned}
E(\eta, \xi, t) = {}& \|\eta(t)\|^2 + \beta\,\|\xi(t)\|^2 + \nu\tau(\sigma+\theta)\,|\eta(t)|_1^2 \\
& + \tau \sum_{t' \leqslant t-\tau} \{r_0\tau(\|\eta_t(t')\|^2 \\
& + \beta\,\|\xi_t(t')\|^2) + \nu(1-6\varepsilon)\,|\eta(t')|_1^2\},
\end{aligned}$$

where $\varepsilon > 0$ is a suitably small constant, $r_0 \geqslant 0$.

Suppose $\mathscr{B}$ is a Banach space, and $\vartheta \subset R^1$ is an interval. We define

$$\mathscr{L}^2(\vartheta, \mathscr{B}) = \left\{\eta/\eta\colon \vartheta \to \mathscr{B},\ \|\eta\|_{\mathscr{L}^2(\vartheta;\mathscr{B})} = \left(\int_\vartheta \|\eta(t)\|_{\mathscr{B}}^2\,dt\right)^{1/2} < \infty\right\}.$$

Similarly, we define the spaces $C(\vartheta; \mathscr{B})$, and $H^\mu(\vartheta; \mathscr{B})$, etc.

For convenience, we recall the definition of non-isotropic Sobolev spaces as [17], for $r \geqslant 0$, and $s \geqslant 0$,

$$H^{r,s}(\Omega) = \mathscr{L}^2(\tilde{I}, H^r(I)) \cap H^s(\tilde{I}, \mathscr{L}^2(I)).$$

equipped with the norm

$$\|\eta\|_{H^{r,s}(\Omega)} = \left(\|\eta\|^2_{\mathscr{L}^2(\tilde{I}, H^r(I))} + \|\eta\|^2_{H^s(\tilde{I}, \mathscr{L}^2(I))}\right)^{1/2}.$$

If $r \geqslant 1$, $s \geqslant 1$, then we define

$$M^{r,s}(\Omega) = H^{r,s}(\Omega) \cap H^1(\tilde{I}, H^{r-1}(I)) \cap H^{s-1}(\tilde{I}, H^1(I)),$$

with the norm

$$\|\eta\|_{M^{r,s}(\Omega)} = \left(\|\eta\|^2_{H^{r,s}(\Omega)} + \|\eta\|^2_{H^1(\tilde{I}, H^{r-1}(I))} + \|\eta\|^2_{H^{s-1}(\tilde{I}, H^1(I))}\right)^{1/2}.$$

Besides, we denote by $H_p^{r,s}(\Omega)$ the closure of $C_p^\infty(\Omega)$ in $H^{r,s}(\Omega)$, $H_{0,p}^{r,s}(\Omega)$, and $M_{0,p}^{r,s}(\Omega)$ are the closures of $C_{0,p}^\infty(\Omega)$ in $H^{r,s}(\Omega)$ and $M^{r,s}(\Omega)$, respectively.

THEOREM 1. *Let (U, P) be the solution of* (2.4) *and (u, p) the solution of* (2.5). *Assume that $U \in C(0, T; M_{0,p}^{r,s}(\Omega)) \cap H^1(0, T; H^1(\Omega) \cap H_p^{r-1,s-1}(\Omega)) \cap H^2(0, T; \mathscr{L}^2(\Omega))$, $P \in C(0, T; H_p^{r-1,s-1}(\Omega)) \cap H^1(0, T; \mathscr{L}^2(\Omega))$, $f \in C(0, T; H_p^{r-1,s-1}(\Omega))$, with $r \geqslant 1$, $s \geqslant 1$. $\bar{r} = \min(r, k+2)$. Suppose $\beta = O(\tau^2)$, $h = O(N^{-\mu})$, and $\tau = O(N^{-\lambda})$ in* (2.5), *$\mu \geqslant 1$, $\lambda > 0$, h, N^{-1}, and τ are sufficiently small. If either of the following two conditions is satisfied*:

(i) $\sigma > \frac{1}{2}$, $\theta > \sigma/(2\sigma - 1)$, *and* $\lambda > \max((1+\mu)/3,\ 1 + 3\mu - 2\mu\bar{r},\ 3 + \mu - 2s)$; (3.1)

(ii) $\sigma \leqslant \frac{1}{2}$ *and* $\tau(C_d h^{-2} + N^2) < 2\theta - 1/(\nu(\sigma + \theta(1 - 2\sigma)))$;

then there exists a positive constant C^* depending only on U, P, f, and ν, such that for all $t \in S_\tau$, we have

$$E(U-u, P-p, t) \leqslant C^*(\tau^2 + h^{2(\bar{r}-1)} + N^{2(1-s)}).$$

Remark 2. If $\delta = \theta > \frac{1}{2}$, then condition (3.1) is not necessary.

IV. NUMERICAL RESULTS

In this section, we examine the numerical performances of the combined spectral-finite element scheme (2.5) by two examples. In each example, the velocity is determined separately by the stream function

(1) $\psi(x, y, t) = 0.1 \exp(x + \sin y + 0.1t)$,

(2) $\psi(x, y, t) = 0.1 \sin(x) \exp(\sin y + 0.1t)$,

while the pressure P is taken to be 0 constantly. Besides, we assume the kinetic viscosity $\nu = 10^{-2}$, and choose the body force f such that U, P, and f satisfy (2.4) exactly.

We consider only the case of $k = 0$ in scheme (2.5); i.e., the finite element subspaces in the x direction for $(u^{(1)}, u^{(2)})$ and p are piecewise linear, piecewise quadratic, and piecewise constant, respectively. For comparison, we also solve (2.4) by the finite element method, in which the domain $\bar{\Omega}$ is divided into $M_h(2N+1)$ congruent small rectangles, each with the length $h_x = 1/M_h$ and the width $h_y = 2\pi/(2N+1)$. We take the trial space for $(u^{(1)}, u^{(2)})$ and p to be piecewise biquadratic, piecewise biquadratic, and piecewise constant, separately. The periodic conditions are also enforced in the y direction. The finite element scheme is constructed similarly to (2.5) by artificial compression. Besides, to avoid calculating the integrations in the nonlinear terms repeatedly when we solve the linear equations in every time level by iteration, we approximate the nonlinear terms explicitly (i.e., $\delta = 0$), in both the spectral-finite element and the finite element schemes, but we always approximate the linear terms implicitly (i.e., $\theta = \sigma = 1$).

For describing the accuracy, we define the discrete L^2-normed relative error for $u^{(1)}$ and $u^{(2)}$ as

$$E^*(u^{(l)}, t) = \left\{ \frac{\sum_{(x,y) \in \Omega^{(l)}} |U^{(l)}(x, y, t) - u^{(l)}(x, y, t)|^2}{\sum_{(x,y) \in \Omega^{(l)}} |U^{(l)}(x, y, t)|^2} \right\}^{1/2},$$

TABLE I

Example (1), $\tau = 10^{-2}$, $\beta = 10^{-3}$, $t = 5$

		$N=4$		$N=8$	
		$E^*(u^{(1)}, t)$	$E^*(u^{(2)}, t)$	$E^*(u^{(1)}, t)$	$E^*(u^{(2)}, t)$
Spectral-F.E.M.	$M_h = 5$	2.1379-3	9.8124-4	7.6151-4	8.2940-4
	$M_h = 10$	1.9619-3	6.0285-4	2.6212-4	3.5686-4
F.E.M.	$M_h = 5$	1.8724-2	1.0882-2	3.8954-3	2.4330-4
	$M_h = 10$	1.6047-2	9.0782-3	3.8383-3	2.2202-3

TABLE II

Example (2), $\tau = 10^{-2}$, $\beta = 10^{-3}$, $t = 5$

		$N=4$		$N=8$	
		$E^*(u^{(1)}, t)$	$E^*(u^{(2)}, t)$	$E^*(u^{(1)}, t)$	$E^*(u^{(2)}, t)$
Spectral-F.E.M.	$M_h = 5$	1.8953-3	6.7098-4	8.9871-4	5.9659-4
	$M_h = 10$	1.7296-3	4.9014-4	5.0291-4	3.5346-4
F.E.M.	$M_h = 5$	1.4452-2	7.6244-3	4.6631-3	2.2337-3
	$M_h = 10$	1.4994-2	7.6532-3	5.7109-3	2.1279-3

TABLE III

Example (1), with Scheme (2.5), $M_h = 5$, $\tau = 10^{-2}$, $t = 5$

	$N=4$		$N=8$	
	$E^*(u^{(1)}, t)$	$E^*(u^{(2)}, t)$	$E^*(u^{(1)}, t)$	$E^*(u^{(2)}, t)$
$\beta = 10^{-2}$	2.1396-3	9.7437-4	7.5778-4	8.2725-4
$\beta = 10^{-3}$	2.1379-3	9.8124-4	7.6151-4	8.2940-4
$\beta = 10^{-4}$	2.0537-3	1.1868-3	9.1537-4	1.1504-3

TABLE IV

Example (2), with Scheme (2.5), $M_h=5$, $\tau=10^{-2}$, $t=5$

	$N=4$		$N=8$	
	$E^*(u^{(1)},t)$	$E^*(u^{(2)},t)$	$E^*(u^{(1)},t)$	$E^*(u^{(2)},t)$
$\beta=10^{-2}$	1.9127-3	6.7048-4	8.9682-4	5.9533-4
$\beta=10^{-3}$	1.8953-3	6.7098-4	8.9871-4	5.9659-4
$\beta=10^{-4}$	1.8107-3	1.0813-3	1.5057-3	1.0408-3

where

$$\Omega^{(1)}=\{(x,y)/x=j_1h_x,\ y=j_2h_y,\quad 1\leqslant j_1\leqslant M_h-1,\ 1\leqslant j_2\leqslant 2N+1\},$$

$$\Omega^{(2)}=\{(x,y)/x=j_1h_x/2,\ y=j_2h_y/2,\quad 1\leqslant j_1\leqslant 2M_h-1,\ 1\leqslant j_2\leqslant 2(2N+1)\}.$$

The numerical results show that

(1) For the same mesh sizes M_h and N, the spectral-finite element scheme gives more accurate results than the finite element scheme does (see Table I and Table II). However, the computing times requires by each are almost the same.

(2) We observe from Theorem 1 that if $\beta=O(\tau^2)$, then the artificial compression term $\beta(\partial P/\partial t)$ does not affect the order of the error $E^*(u^{(l)},t)$. However, in practical computation, the value of β determines the structure of the linear equations derived from scheme (2.5). An excessively small β would result in a large condition number of the corresponding matrix (in fact, the matrix tends to be singular, as β tends to 0) and require a greater amount of computation. Consequently, we should take into account the matrix structure apart from the approximation order, when we select an appropriate β. In the examples in this section, the results for $\beta=10^{-2}$ and 10^{-3} are better than that for $\beta=10^{-4}$ (see Table III and Table IV).

V. SOME LEMMAS

To prove Theorem 1, we need the following lemmas.

LEMMA 1. [8]. *If $r>\frac{1}{2}$, $s\geqslant 0$, $\bar r=\min(r,k+1)$, then there exists a positive constant c independent of h and N, such that for all $\eta\in H_p^{r,s}(\Omega)$,*

$$\|\eta-P_N\circ\Pi_h^k\eta\|\leqslant c(h^{\bar r}+N^{-s})\,\|\eta\|_{H^{\bar r,s}(\Omega)}.$$

LEMMA 2 [8]. *If $Nh\leqslant$ const, $r\geqslant 1$, $s\geqslant 1$, $\bar r=\min(r,k+1)$, then there exists a positive constant c independent of h and N, such that for all $\eta\in M_p^{r,s}(\Omega)$,*

$$|\eta-P_N\circ\Pi_h^k\eta|_1\leqslant c(h^{\bar r-1}+N^{1-s})\,\|\eta\|_{M^{\bar r,s}(\Omega)}.$$

In particular, we have for all $\eta\in H_p^1(\Omega)$ that

$$|P_N\circ\Pi_h^k\eta|_1\leqslant c\,\|\eta\|_1.$$

LEMMA 3. *There exists a linear operator Q_α: $[H^1_{0,P}(\Omega)]^2\to X_\alpha$, and a positive constant c independent of h and N, such that*

(i) $b(\eta-Q_\alpha\eta,\omega)=0,\quad \forall\omega\in Y_\alpha,\ \forall\eta\in[H^1_{0,P}(\Omega)]^2.$ (5.1)

(ii) *If $Nh\leqslant$ Const, and $r\geqslant 1$, $s\geqslant 1$, then we have for all $\eta\in[M^{r,s}(\Omega)]^2$ that*

$$|\eta-Q_\alpha\eta|_1\leqslant c(h^{\bar r-1}+N^{1-s})\,\|\eta\|_{M^{\bar r,s}(\Omega)},\quad \bar r=\min(r,k+2),\tag{5.2}$$

$$|Q_\alpha\eta|_1\leqslant c\,|\eta|_1.\tag{5.3}$$

Proof. Let

$$\eta=\begin{pmatrix}\eta^{(1)}\\ \eta^{(2)}\end{pmatrix}=\sum_{|j|=0}^{\infty}\begin{pmatrix}\eta_j^{(1)}(x)\\ \eta_j^{(2)}(x)\end{pmatrix}e^{ijy}\in(H^1_{0,P}(\Omega))^2.$$

We have from Sobolev's embedding theorem that $H^1_{0,P}(\Omega)\subsetneq\mathscr{L}^2(\tilde I,C(I))$. Hence

$$\eta_j^{(m)}\in C(I),\qquad \forall\,|j|<\infty,\quad m=1,2.$$

Let

$$Q_\alpha\eta=\begin{pmatrix}Q_\alpha^{(1)}\eta^{(1)}\\ Q_\alpha^{(2)}\eta^{(2)}\end{pmatrix}=\sum_{|j|\leqslant N}\begin{pmatrix}Q_h^{(1)}\eta_j^{(1)}(x)\\ Q_h^{(2)}\eta_j^{(2)}(x)\end{pmatrix}e^{ijy},$$

where $Q_h^{(m)}\eta_j^{(m)}\in S_h^{k+m}$, $m=1,2$, are defined by

$$Q_h^{(m)}\eta_j^{(m)}(x_l)=\eta_j^{(m)}(x_l),\qquad 0\leqslant l\leqslant M_h.$$

$$\int_{I_l}[Q_h^{(m)}\eta_j^{(m)}(x)-\eta_j^{(m)}(x)]\,\omega(x)\,dx=0,\tag{5.4}$$

$$\forall\omega\in P_{k+m-2},\ 1\leqslant l\leqslant M_h.$$

It is obvious that Q_α is a well-defined linear operator from $[H^1_{0,P}(\Omega)]^2$ onto X_α.

We first prove (5.1). Suppose

$$\omega=\sum_{|j|\leqslant N}\omega_j(x)\,e^{ijy}\in Y_\alpha.$$

Then we have $\omega_j \in \tilde{S}_h^k$, and hence $\omega_j|_{I_l} \in P_k$. Thus we have from (5.4) that

$$\begin{aligned} b(\eta - Q_\alpha \eta, \omega) &= \sum_{|j| \leqslant N} \int_I \left[\frac{\partial}{\partial x}(\eta_j^{(1)} - Q_h^{(1)} \eta_j^{(1)})(x) + ij(\eta_j^{(2)} - Q_h^{(2)} \eta_j^{(2)})(x)\right] \omega_j(x)\, dx \\ &= \sum_{|j| \leqslant N} \sum_{1 \leqslant l \leqslant M_h} \int_{I_l} \left[-(\eta_j^{(1)}(x) - Q_h^{(1)} \eta_j^{(1)}(x)) \times \frac{\partial \omega_j}{\partial x}(x) + ij(\eta_j^{(2)}(x) - Q_h^{(2)} \eta_j^{(2)}(x)\, \omega(x)\right] dx \\ &= 0. \end{aligned}$$

Next we prove (5.2). Consider the reference element $\hat{I} = [0, 1]$, and a linear operator $\hat{Q}: C(\hat{I}) \to P_{k'+2}$ ($k' = k$ or $k-1$) defined as: for any $\hat{\xi} \in C(\hat{I})$, $\hat{Q}\hat{\xi} \in P_{k'+2}$ satisfies

$$\begin{aligned} &\hat{Q}\hat{\xi}(x) = \hat{\xi}(x), \qquad x = 0, 1, \\ &\int_{\hat{I}} (\hat{Q}\hat{\xi}(x) - \hat{\xi}(x))\, \omega(x)\, dx = 0, \qquad \forall \omega \in P_{k'}. \end{aligned} \tag{5.5}$$

It is not difficult to verify that

$$\hat{Q} \in \mathscr{L}(C(\hat{I}), W^{m,q}(\hat{I})), \qquad \forall m \geqslant 0, q \geqslant 1, \tag{5.5}$$

and

$$\hat{Q}\hat{\xi} = \hat{\xi}, \qquad \forall \hat{\xi} \in P_{k'+2}. \tag{5.6}$$

Moreover, we define the operator $F: \hat{I} \to I_l$ by

$$x = F(\hat{x}) = h_l \hat{x} + x_{l-1}.$$

and a correspondence between $C(\hat{I})$ and $C(\bar{I}_l)$, i.e., for any $\xi \in C(\bar{I}_l)$,

$$\hat{\xi}(\hat{x}) = \hat{\xi}(F^{-1}(x)) = \xi(x).$$

Then we can prove for any $Q_{h,l}$, the restriction of $Q_h^{(m)}$ ($m = 1$ or 2) on I_l, that

$$\widehat{Q_{h,l}\xi} = \hat{Q}\hat{\xi}. \tag{5.7}$$

In fact,

$$\widehat{Q_{h,l}\xi}(0) = Q_{h,l}\xi(x_{l-1}) = \xi(x_{l-1}) = \hat{\xi}(0) = \hat{Q}\hat{\xi}(0),$$

and, similarly, $\widehat{Q_{h,l}\xi}(1) = \hat{Q}\hat{\xi}(1)$. Besides, we have

$$\begin{aligned} \int_{\hat{I}} (\widehat{Q_{h,l}\xi} - \hat{\xi})(\hat{x})\, \hat{\omega}(\hat{x})\, d\hat{x} &= \frac{1}{h_l} \int_{I_l} (Q_{h,l}\xi - \xi)(x)\, \omega(x)\, dx \\ &= 0, \qquad \forall \hat{\omega} \in P_{k'}. \end{aligned}$$

The uniqueness of $\hat{Q}$ assures the validity of (5.7).

By an argument analogous to the classical error estimation for function interpolation (see [18]), we obtain from (5.5)–(5.7) that

$$\|\eta_j^{(m)} - Q_h^{(m)} \eta_j^{(m)}\| \leqslant c h^{\bar{r}_m} |\eta_j^{(m)}|_{H^{\bar{r}_m}(I)},$$

$$|\eta_j^{(m)} - Q_h^{(m)} \eta_j^{(m)}|_1 \leqslant c h^{\bar{r}_m - 1} |\eta_j^{(m)}|_{H^{\bar{r}_m}(I)},$$

where $\bar{r}_m = \min(r, k + m + 1)$. Thus we have

$$\begin{aligned} |\eta - Q_\alpha \eta|_1^2 &= \sum_{m=1}^{2} \left\{ \sum_{|j| \leqslant N} [|\eta_j^{(m)} - Q_j^{(m)} \eta_j^{(m)}|_1^2 + j^2 \|\eta_j^{(m)} - Q_h^{(m)} \eta_j^{(m)}\|^2] + \sum_{|j| > N} [|\eta_j^{(m)}|_1^2 + j^2 \|\eta_j^{(m)}\|^2] \right\} \\ &\leqslant \sum_{m=1}^{2} \left\{ c h^{2(\bar{r}-1)} \sum_{|j| \leqslant N} [|\eta_j^{(m)}|_{H^{\bar{r}}(I)}^2 + h^2 j^2 |\eta_j^{(m)}|_{H^{\bar{r}}(I)}^2] + c N^{2(1-s)} \sum_{|j| > N} [|j|^{2(s-1)} |\eta_j^{(m)}|_1^2 + |j|^{2s} \|\eta_j^{(m)}\|^2] \right\} \\ &\leqslant c(h^{2(\bar{r}-1)} + N^{2(1-s)}) \|\eta\|_{M^{\bar{r},s}(\Omega)}^2, \end{aligned}$$

with $\bar{r} = \min(\bar{r}_1, \bar{r}_2) = \min(r, k+2)$.

Finally, we can obtain (5.3) by putting $r = s = 1$ in (5.2).

LEMMA 4. *There exists a positive constant C_d, depending on the parameter d, but not on h and N, such that for all $\eta \in S_h^{k+m} \otimes S_N$, $m = 1, 2$,*

$$\|\eta\|_{L^\infty(\Omega)} \leqslant c_d h^{-1/2} N^{1/2} \|\eta\|, \tag{5.8}$$

$$|\eta|_1 \leqslant (c_d h^{-2} + N^2)^{1/2} \|\eta\|. \tag{5.9}$$

Proof. Let

$$\eta = \sum_{|j| \leqslant N} \eta_j(x)\, e^{ijy} \in S_h^{k+m} \otimes S_N;$$

then $\eta_j \in S_h^{k+m}$. Thus by the inverse inequality in the finite element method, we have [18]

$$\|\eta_j\|_{L^\infty(I)} \leqslant c_d h^{-1/2} \|\eta_j\|,$$
$$|\eta_j|_1 \leqslant c_d h^{-1} \|\eta_j\|,$$

and hence

$$\begin{aligned}\|\eta\|_{L^\infty(\Omega)} &\leqslant \sum_{|j| \leqslant N} |\eta_j|_{L^\infty(I)} \\ &\leqslant c_d h^{-1/2} \sum_{|j| \leqslant N} \|\eta_j\| \\ &\leqslant c_d h^{-1/2} N^{1/2} \left(\sum_{|j| \leqslant N} \|\eta_j\|^2 \right)^{1/2} \\ &\leqslant c_d h^{-1/2} N^{1/2} \|\eta\|,\end{aligned}$$

$$\begin{aligned}|\eta|_1 &= \left[\sum_{|j| \leqslant N} (|\eta_j|_1^2 + j^2 \|\eta_j\|^2) \right]^{1/2} \\ &\leqslant \left[\sum_{|j| \leqslant N} (c_d h^{-2} \|\eta_j\|^2 + N^2 \|\eta_j\|^2) \right]^{1/2} \\ &\leqslant (c_d h^{-2} + N^2)^{1/2} \|\eta\|.\end{aligned}$$

LEMMA 5. *There exists a positive constant c depending only on Ω, such that for all* $\eta, \xi, \varphi \in [H_{0,P}^1(\Omega)]^2$,

$$|J(\eta, \varphi, \xi)| \leqslant c\, |\eta|_1\, |\xi|_1\, \|\varphi\|^{1/2}\, |\varphi|_1^{1/2}, \tag{5.10}$$

$$|J(\eta, \varphi, \xi)| \leqslant c\, |\varphi|_1 (|\eta|_1 \|\xi\|^{1/2} |\xi|_1^{1/2} + \|\eta\|^{1/2} |\eta|_1^{1/2} |\xi|_1), \tag{5.11}$$

$$|J(\eta, \varphi, \xi)| \leqslant c\, |\eta|_1\, |\xi|_1\, |\varphi|_1. \tag{5.12}$$

Proof. We have from Sobolev's embedding theorem that $H^1(\Omega) \subsetneq \mathscr{L}^6(\Omega)$. By the equivalence of the norm $\|\cdot\|_1$ and the semi-norm $|\cdot|_1$ over $H_{0,P}^1(\Omega)$, and Hölder's inequality, we have

$$\|\varphi\|_{\mathscr{L}^3(\Omega)} \leqslant \|\varphi\|_{\mathscr{L}^2(\Omega)}^{1/2} \|\varphi\|_{\mathscr{L}^6(\Omega)}^{1/2} \leqslant c\, \|\varphi\|^{1/2}\, |\varphi|_1^{1/2}.$$

Hence

$$\begin{aligned}|J(\eta, \varphi, \xi)| &\leqslant \tfrac{1}{2} \int_\Omega |((\varphi \cdot \nabla)\eta)\xi|\, d\Omega \\ &\quad + \tfrac{1}{2} \int_\Omega |((\varphi \cdot \nabla)\xi)\eta|\, d\Omega \\ &\leqslant c\, |\eta|_1\, \|\xi\|_{\mathscr{L}^6(\Omega)}\, \|\varphi\|_{\mathscr{L}^3(\Omega)} \\ &\quad + c\, |\xi|_1\, \|\eta\|_{\mathscr{L}^6(\Omega)}\, \|\varphi\|_{\mathscr{L}^3(\Omega)} \\ &\leqslant c\, |\eta|_1\, |\xi|_1\, \|\varphi\|^{1/2}\, |\varphi|_1^{1/2}.\end{aligned}$$

Inequality (5.11) can be proved similarly; (5.12) can be obtained readily by (5.10) and $\|\varphi\| \leqslant c\, |\varphi|_1$.

LEMMA 6 [19]. *Suppose that the following conditions are fulfilled,*

(i) *Z(t) is a non-negative function defined on* S_τ, D_1, D_2, *and* ρ *are non-negative constants,*

(ii) *H is a real-valued function defined on* R^1, *such that* $H(\xi) \leqslant 0$ *for* $\xi \leqslant D_2$,

(iii) *for all* $t \in S_\tau$,

$$Z(t) \leqslant \rho + \sum_{t' \leqslant t-\tau} [D_1 Z(t') + H(Z(t'))],$$

(iv) $Z(0) \leqslant \rho$, *and* $\rho e^{D_1 t_1} \leqslant D_2$, *for some* $t_1 \in S_\tau$; *then, we have for all* $t \leqslant t_1$ *that*

$$Z(t) \leqslant \rho e^{D_1 t}.$$

In particular, if $H(\xi) \leqslant 0$ *for all* $\xi \in R^1$, *then the above inequality holds for all* $t \in S_\tau$ *and any* ρ.

VI. THE PROOF OF CONVERGENCE

Let the pair (U, P) be the solution of (2.4). Assume that they are suitably smooth. Let

$$u^*(t) = Q_\alpha U(t), \qquad p^*(t) = \varphi_\alpha P(t).$$

Then we have from (2.4) and (5.1) that

$$\begin{aligned}&(u_t^*(t), v) + J(u^*(t) + \delta\tau u_t^*(t), u^*(t), v) \\ &\quad - b(v, p^*(t) + \theta\tau p_t^*(t)) \\ &\quad + \nu a(u^*(t) + \sigma\tau u_t^*(t), v) \\ &\quad = (P_N \circ \Pi_h^k f(t), v) + \sum_{l=1}^{8} E_l(v), \\ &\quad \forall v \in (H_{0,P}^1(\Omega))^2, \\ &\beta(p_t^*(t), \omega) + b(u^*(t) + \theta\tau u_t^*(t), \omega) \\ &\quad = E_9(w), \qquad \forall \omega \in \tilde{\mathscr{L}}^2(\Omega), \\ &u^*(0) = Q_\alpha U_0, \qquad p^*(0) = \varphi_\alpha P(0),\end{aligned} \tag{6.1}$$

where

$$E_1(v) = \left(u_t^* - \frac{\partial U}{\partial t}, v \right),$$
$$E_2(v) = J(u^*, u^*, v) - J(U, U, v),$$
$$E_3(v) = \delta\tau J(u^*, u^*, v),$$
$$E_4(v) = b(v, P - p^*),$$

$$E_5(v) = -\theta\tau b(v, p_t^*),$$
$$E_6(v) = \nu a(u^* - U, v),$$
$$E_7(v) = \nu\sigma\tau a(u_t^*, v),$$
$$E_8(v) = (f - P_N \circ \Pi_h^k f, v),$$
$$E_9(\omega) = \beta(p_t^*, \omega).$$

Let the pair (u, p) be the solution of (2.5). Define

$$\tilde{u}(t) = u(t) - u^*(t), \qquad \tilde{p}(t) = p(t) - p^*(t).$$

By substracting (2.5) from (6.1), we obtain that

$$\begin{aligned}
&(\tilde{u}_t(t), v) + J(u^*(t) + \delta\tau u_t^*(t), \tilde{u}(t), v) \\
&\quad + J(\tilde{u}(t) + \delta\tau\tilde{u}_t(t), u^*(t) + \tilde{u}(t), v) \\
&\quad - b(v, \tilde{p}(t) + \theta\tau\tilde{p}_t(t)) \\
&\quad + \nu a(\tilde{u}(t) + \sigma\tau\tilde{u}_t(t), v) \\
&= -\sum_{l=1}^{8} E_l(v), \quad \forall v \in X_\alpha, \qquad (6.2)\\
&\beta(\tilde{p}_t(t), \omega) + b(\tilde{u}(t) + \theta\tau\tilde{u}_t(t), \omega) \\
&\quad = -E_9(\omega), \qquad \forall \omega \in Y_\alpha, \\
&\quad \tilde{u}(0) = -Q_\alpha U_0 + P_N \circ \Pi_h^{k+1} U_0, \\
&\quad \tilde{p}(0) = -\varphi_\alpha P(0).
\end{aligned}$$

Let $m > 1$ be a undetermined constant. Noting (3.1) and the identity [19]

$$2(\tilde{u}_t(t), \tilde{u}(t)) = (\|\tilde{u}(t)\|^2)_t - \tau\,\|\tilde{u}_t(t)\|^2, \qquad (6.3)$$

we have by taking $v = 2\tilde{u}(t) + m\tau\tilde{u}_t(t)$ in the first equation of (6.2) that

$$\begin{aligned}
&(\|\tilde{u}(t)\|^2)_t + \tau(m-1)\|\tilde{u}_t(t)\|^2 + 2\nu\,|\tilde{u}(t)|_1^2 \\
&\quad + \nu\tau\left(\sigma + \frac{m}{2}\right)(|\tilde{u}(t)|_1^2)_t \\
&\quad + \nu\tau^2\left(m\sigma - \sigma - \frac{m}{2}\right)|\tilde{u}_t(t)|_1^2 \\
&\quad + \sum_{l=1}^{4} F_l(t) - b(2\tilde{u}(t) + m\tau\tilde{u}_t(t), \tilde{p}(t) + \theta\tau\tilde{p}_t(t)) \\
&= -\sum_{l=1}^{8} E_l(2\tilde{u}(t) + m\tau\tilde{u}_t(t)), \qquad (6.4)
\end{aligned}$$

where

$$F_1 = 2J(u^* + \delta\tau u_t^*, \tilde{u}, \tilde{u}),$$
$$F_2 = m\tau J(u^* + \delta\tau u_t^*, \tilde{u}, \tilde{u}_t),$$
$$F_3 = \tau(m - 2\delta)\,J(\tilde{u}, u^*, \tilde{u}_t),$$
$$F_4 = \tau(m - 2\delta)\,J(\tilde{u}, \tilde{u}, \tilde{u}_t).$$

By taking $\omega = 2\tilde{p}(t) + m\tau\tilde{p}_t(t)$ in the second equation of (6.2), we have from (6.3) that

$$\begin{aligned}
&\beta(\|\tilde{p}(t)\|^2)_t + \beta\tau(m-1)\,\|\tilde{p}_t(t)\|^2 \\
&\quad + b(\tilde{u}(t) + \theta\tau\tilde{u}_t(t), 2\tilde{p}(t) + m\tau\tilde{p}_t(t)) \\
&= -E_9(2\tilde{p}(t) + m\tau\tilde{p}_t(t)). \qquad (6.5)
\end{aligned}$$

By putting (6.4) and (6.5) together, we obtain

$$\begin{aligned}
&(\|\tilde{u}(t)\|^2 + \beta\,\|\tilde{p}(t)\|^2)_t + \tau(m-1)\,(\|\tilde{u}_t(t)\|^2 \\
&\quad + \beta\,\|\tilde{p}_t(t)\|^2) + 2\nu\,|\tilde{u}(t)|_1^2 \\
&\quad + \nu\tau\left(\sigma + \frac{m}{2}\right)(|\tilde{u}(t)|_1^2)_t \\
&\quad + \nu\tau^2\left(m\sigma - \sigma - \frac{m}{2}\right)|\tilde{u}_t(t)|_1^2 \\
&\quad + \sum_{l=1}^{4} F_l(t) + \tau(2\theta - m)\,H(t) \\
&= -\sum_{l=1}^{8} E_l(2\tilde{u}(t) + m\tau\tilde{u}_t(t)) \\
&\quad - E_9(2\tilde{p}(t) + m\tau\tilde{p}_t(t)), \qquad (6.6)
\end{aligned}$$

where

$$H(t) = b(\tilde{u}_t, \tilde{p}) - b(\tilde{u}, \tilde{p}_t).$$

We now estimate $|F_l|$. Suppose $\varepsilon > 0$ is a suitably small constant. Then we have from (5.3) and (5.10) that

$$\begin{aligned}
|F_1| &\leq c\,|u^* + \delta\tau u_t^*|_1\,\|\tilde{u}\|^{1/2}\,|\tilde{u}|_1^{3/2} \\
&\leq c\,\|U\|_{C(0,T;H^1(\Omega))}\,\|\tilde{u}\|^{1/2}\,|\tilde{u}|_1^{3/2} \\
&\leq \frac{1}{2}\varepsilon\nu\,|\tilde{u}|_1^2 \\
&\quad + \frac{c}{\varepsilon\nu}\,\|U\|^2_{C(0,T;H^1(\Omega))}\,\|\tilde{u}\|\,|\tilde{u}|_1 \\
&\leq \varepsilon\nu\,|\tilde{u}|_1^2 + \frac{C}{\varepsilon^3\nu^3}\,\|U\|^4_{C(0,T;H^1(\Omega))}\,\|\tilde{u}\|^2,
\end{aligned}$$

and

$$|F_2| \leqslant cm\tau \, \|U\|_{C(0,T;H^1(\Omega))} \times \|\tilde{u}\|^{1/2} \, |\tilde{u}|_1^{1/2} \, |\tilde{u}_t|_1 \leqslant \varepsilon\nu\tau \, |\tilde{u}|_1 \, |\tilde{u}_t|_1 + \frac{c\tau m^2}{\varepsilon\nu} \|U\|^2_{C(0,T;H^1(\Omega))} \|\tilde{u}\| \, |\tilde{u}_t|_1 \leqslant \varepsilon\nu \, |\tilde{u}|_1^2 + \varepsilon\nu\tau^2 \, |\tilde{u}_t|_1^2 + \frac{cm^4}{\varepsilon^3\nu^3} \|U\|^4_{C(0,T;H^1(\Omega))} \|\tilde{u}\|^2.$$

Similarly, we can obtain from (5.11) that

$$|F_3| \leqslant c\tau \, |m-2\delta| \, |u^*|_1 \times (|\tilde{u}|_1 \, \|\tilde{u}_t\|^{1/2} \, |\tilde{u}_t|_1^{1/2} + \|\tilde{u}\|^{1/2} \, |\tilde{u}|_1^{1/2} \, |\tilde{u}_t|_1) \leqslant \varepsilon\nu\tau \, |\tilde{u}|_1 \, |\tilde{u}_t|_1 + \frac{c\tau(m-2\delta)^2}{\varepsilon\nu} \|U\|^2_{C(0,T;H^1(\Omega))} \times (\|\tilde{u}\| \, |\tilde{u}_t|_1 + |\tilde{u}|_1 \, \|\tilde{u}_t\|) \leqslant \varepsilon\nu \, |\tilde{u}|_1^2 + \varepsilon\nu\tau^2 \, |\tilde{u}_t|_1^2 + \varepsilon\tau \, \|\tilde{u}_t\|^2 + \frac{c(m-2\delta)^4}{\varepsilon^3\nu^3} \times \|U\|^4_{C(0,T;H^1(\Omega))} (\|\tilde{u}\|^2 + \tau \, |\tilde{u}|_1^2).$$

From integrating by parts in F_4, we have by (5.8) that

$$|F_4| = \tau \, |m-2\delta| \left| \int_\Omega [((\tilde{u}\cdot\nabla)\tilde{u}) \, \tilde{u}_t + \frac{1}{2}(\nabla\cdot\tilde{u})(\tilde{u}\cdot\tilde{u}_t)] \, d\Omega \right| \leqslant c\tau \, |m-2\delta| \, \|\tilde{u}\|_{\mathscr{L}^\infty(\Omega)} \, |\tilde{u}|_1 \, \|\tilde{u}_t\| \leqslant \varepsilon\tau \, \|\tilde{u}_t\|^2 + \frac{c\tau N(m-2\delta)^2}{\varepsilon h} \|\tilde{u}\|^2 \, |\tilde{u}|_1^2.$$

Next we examine $|E_l|$. First, we have

$$|E_1(2\tilde{u}+m\tau\tilde{u}_t)| \leqslant \|\tilde{u}\|^2 + \varepsilon\tau \, \|\tilde{u}_t\|^2 + \left(1+\frac{\tau m^2}{4\varepsilon}\right) \left\| u_t^* - \frac{\partial U}{\partial t} \right\|^2.$$

We have, furthermore, from (5.12) that

$$|E_2(2\tilde{u}+m\tau\tilde{u}_t)| \leqslant |J(U, u^*-U, 2\tilde{u}+m\tau\tilde{u}_t)| + |J(u^*-U, u^*, 2\tilde{u}+m\tau\tilde{u}_t)| \leqslant c \, |U|_1 \, |u^*-U|_1 \, |2\tilde{u}+m\tau\tilde{u}_t|_1 \leqslant \varepsilon\nu \, |\tilde{u}|_1^2 + \varepsilon\nu\tau^2 \, |\tilde{u}_t|_1^2 + \frac{c}{\varepsilon\nu}\left(1+\frac{m^2}{4}\right) \times \|U\|^2_{C(0,T;H^1(\Omega))} \, |u^*-U|_1^2$$

and

$$|E_3(2\tilde{u}+m\tau\tilde{u}_t)| \leqslant c\delta\tau \, |u_t^*|_1 \, |\tilde{u}|_1 \, |2\tilde{u}+m\tau\tilde{u}_t|_1 \leqslant \varepsilon\nu \, |\tilde{u}|_1^2 + \varepsilon\nu\tau^2 \, |\tilde{u}_t|_1^2 + \frac{c\delta^2\tau^2}{\varepsilon\nu}\left(1+\frac{m^4}{4}\right) |U_t|_1^2 \, |\tilde{u}|_1^2.$$

Besides, it is not difficult to show that

$$\sum_{l=4}^{7} |E_l(2\tilde{u}+m\tau\tilde{u}_t)| \leqslant \varepsilon\nu \, |\tilde{u}|_1^2 + \varepsilon\nu\tau^2 \, |\tilde{u}_t|_1^2 + \frac{c}{\varepsilon\nu}\left(1+\frac{m^4}{4}\right) (\|p^*-P\|^2 + \nu \, |u^*-U|_1^2 + \theta^2\tau^2 \, \|p_t^*\|^2 + \nu\sigma^2\tau^2 \, |U_t|_1^2),$$

$$|E_8(2\tilde{u}+m\tau\tilde{u}_t)| \leqslant \|\tilde{u}\|^2 + \varepsilon\tau \, \|\tilde{u}_t\|^2 + \left(1+\frac{\tau m^2}{4\varepsilon}\right) \|f - P_N \circ \Pi_h^k f\|^2,$$

$$|E_9(2\tilde{p}+m\tau\tilde{p}_t)| \leqslant \beta \, \|\tilde{p}\|^2 + \varepsilon\beta\tau \, \|\tilde{p}_t\|^2 + \beta\left(1+\frac{\tau m^2}{4\varepsilon}\right) \|p_t^*\|^2.$$

By substituting the above estimations into (6.6), we obtain

$$(\|\tilde{u}(t)\|^2 + \beta \, \|\tilde{p}(t)\|^2)_t + \tau(m-1-5\varepsilon)(\|\tilde{u}_t(t)\|^2 + \beta \, \|\tilde{p}_t(t)\|^2) + \nu(1-6\varepsilon) \, |\tilde{u}(t)|_1^2 + \nu\tau\left(\sigma+\frac{m}{2}\right)(|\tilde{u}(t)|_1^2)_t + \nu\tau^2\left(m\sigma-\sigma-\frac{m}{2}-5\varepsilon\right) |\tilde{u}_t(t)|_1^2 + \tau(2\theta-m) \, H(t) \leqslant C_1 \, (\|\tilde{u}(t)\|^2 + \beta \, \|\tilde{p}(t)\|^2) + C_2(t) \, |\tilde{u}(t)|_1^2 + G(t), \tag{6.7}$$

where

$$C_1 = 2 + \frac{c(1+m^4+(m-2\delta)^4)}{\varepsilon^3 v^3} \|U\|^4_{C(0,T;H^1(\Omega))},$$

$$C_2(t) = -v + \frac{c\tau(m-2\delta)^4}{\varepsilon^3 v^3} \|U\|^4_{C(0,T;H^1(\Omega))} + \frac{c\delta^2\tau^2}{\varepsilon v}\left(1+\frac{m^2}{4}\right)\|U_t(t)\|^2 + \frac{c\tau N(m-2\delta)^2}{\varepsilon h}\|\tilde{u}(t)\|^2,$$

$$G(t) = \left(1+\frac{\tau m^2}{4\varepsilon}\right)\left(\left\|u_t^* - \frac{\partial U}{\partial t}\right\|^2 + \beta\|p_t^*(t)\|^2 + \|f(t) - P_N \circ \Pi_h^k f(t)\|^2\right) + \frac{c}{\varepsilon v}\left(1+\frac{m^4}{4}\right)[(v+\|U\|^2_{C(0,T;H^1(\Omega))}) \times |u^*(t)-U(t)|_1^2 + \|p_t^*(t) - P(t)\|^2 + v\sigma^2\tau^2 |U_t(t)|_1^2 + \theta^2\tau^2 \|p_t^*(t)\|^2].$$

Now we choose the constants m and ε. Take $m = 2\theta$ and $r_0 \geqslant 0$ to be sufficiently small. If $\sigma > \frac{1}{2}$ and $\theta > \sigma/(2\sigma-1)$, then we can take ε and r_0 to be so small that

$$2\theta \geqslant \max\left(1+4\varepsilon+r_0, \frac{2\sigma+10\varepsilon}{2\sigma-1}\right).$$

If $\sigma \leqslant \theta/(2\theta-1)$, and

$$v\tau(C_d h^{-2} + N^2) < \frac{2\theta-1}{\sigma+\theta(1-2\sigma)}, \tag{6.8}$$

then we take ε and r_0 to be so small that

$$2\theta - 1 - 4\varepsilon - r_0 \geqslant v\tau[2\theta(\tfrac{1}{2}-\sigma)+\sigma+5\varepsilon] \times (C_d h^{-2} + N^2).$$

By (5.9), we have in both cases that

$$\tau(m-1-4\varepsilon)(\|\tilde{u}_t(t)\|^2 + \beta\|\tilde{p}_t(t)\|^2) + v\tau^2\left(m\sigma - \sigma - \frac{m}{2} - 5\varepsilon\right)|\tilde{u}_t(t)|_1^2 \geqslant r_0\tau(\|\tilde{u}_t(t)\|^2 + \beta\|\tilde{p}_t(t)\|^2).$$

Consequently, we obtain from (6.7) that

$$(\|\tilde{u}(t)\|^2 + \beta\|\tilde{p}(t)\|^2)_t + r_0\tau(\|\tilde{u}_t(t)\|^2 + \beta\|\tilde{p}_t(t)\|^2) + v(1-6\varepsilon)|\tilde{u}(t)|_1^2 + v\tau\left(\sigma+\frac{m}{2}\right)(|\tilde{u}(t)|_1^2)_t \leqslant C_1(\|\tilde{u}(t)\|^2 + \beta\|\tilde{p}(t)\|^2) + C_2(t)|\tilde{u}(t)|_1^2 + G(t). \tag{6.9}$$

Let $E(\eta, \xi, t)$ be defined as in Section III, and

$$\rho(t) = \|\tilde{u}(0)\|^2 + \beta\|\tilde{p}(0)\|^2 + v\tau\left(\sigma+\frac{m}{2}\right)|\tilde{u}(0)|_1^2 + \tau\sum_{t' \leqslant t-\tau} G(t').$$

By summing (6.9) for all $t' \leqslant t-\tau$, $t' \in S_\tau$, we obtain

$$E(\tilde{u}, \tilde{p}, t) \leqslant \rho(t) + \tau\sum_{t' \leqslant t-\tau}(C_1 E(\tilde{u}, \tilde{p}, t') + C_2(t')|\tilde{u}(t')|_1^2). \tag{6.10}$$

Clearly, if the mesh size τ is sufficiently small, then we have

$$C_2(t) \leqslant -\frac{v}{2} + \frac{c\tau N(m-2\delta)^2}{\varepsilon h}\|\tilde{u}(t)\|^2.$$

Hence we have from Lemma 6 that if there exists a $t_1 \in S_\tau$ such that

$$\rho(t_1) \leqslant \frac{\varepsilon v h}{2c\tau N(m-2\delta)^2}, \tag{6.11}$$

then we have for all $t \leqslant t_1$, $t \in S_\tau$,

$$E(\tilde{u}, \tilde{p}, t) \leqslant \rho(t)\, e^{C_1 t}.$$

Thus, in order to obtain the convergence, we only need to obtain the order of $\rho(t)$ and verify (6.11).

By Lemma 2 and (5.2), we have

$$\|\tilde{u}(0)\| \leqslant c\,|\tilde{u}(0)|_1 \leqslant c(|U(0) - Q_\alpha U(0)|_1 + |U(0) - P_N \circ \Pi_h^{k+1} U(0)|_1) \leqslant c(h^{\bar{r}-1} + N^{1-s})\,|U(0)|_{M^{\bar{r},s}(\Omega)},$$

$$\bar{r} = \min(r, k+2),$$

$$|u^*(t) - U(t)|_1 \leqslant c(h^{\bar{r}-1} + N^{1-s})\,\|U(t)\|_{M^{\bar{r},s}(\Omega)}.$$

By Lemma 1, we have

$$\|\tilde{p}(0)\| \leqslant c\, \|P(0)\|,$$
$$\|p^*(t)-P(t)\| \leqslant c(h^{\tilde{r}-1}+N^{1-s}) \times \|P(t)\|_{H^{\tilde{r}-1,s-1}(\Omega)},$$
$$\|f(t)-P_N \circ \Pi_h^k f(t)\| \leqslant c(h^{\tilde{r}-1}+N^{1-s}) \times \|f(t)\|_{H^{\tilde{r}+1,s-1}(\Omega)}.$$

By using Taylor's formula, we obtain that

$$|U_t(t)|_1 = \frac{1}{\tau}\left|\int_t^{t+\tau} \frac{\partial U}{\partial t}(t')\,dt'\right|_1 \leqslant \tau^{-1/2}\left(\int_t^{t+\tau}\left|\frac{\partial U}{\partial t}(t')\right|_1^2 dt'\right)^{1/2},$$

and, similarly,

$$\|p_t^*(t)\| \leqslant c\,\|P_t(t)\| \leqslant c\tau^{-1/2}\left(\int_t^{t+\tau}\left\|\frac{\partial P}{\partial t}(t')\right\|^2 dt'\right)^{1/2}.$$

Besides, it is not difficult to show that

$$\begin{aligned}\left\|u_t^*(t)-\frac{\partial U}{\partial t}(t)\right\| &\leqslant \|U_t(t)-u_t^*(t)\| + \left\|U_t(t)-\frac{\partial U}{\partial t}(t)\right\| \\ &\leqslant \frac{1}{\tau}\int_t^{t+\tau}\left\|\frac{\partial}{\partial t}(U-u^*)(t')\right\| dt' + \frac{1}{\tau}\int_t^{t+\tau}\int_t^{t'}\left\|\frac{\partial^2 U}{\partial t^2}(t'')\right\| dt''\,dt' \\ &\leqslant c\tau^{-1/2}(h^{\tilde{r}-1}+N^{1-s}) \times \left[\int_t^{t+\tau}\left\|\frac{\partial U}{\partial t}(t')\right\|_{H^{\tilde{r}-1,s-1}(\Omega)}^2 dt'\right]^{1/2} \\ &\quad + c\tau^{-1/2}\left(\int_\tau^{t+\tau}\left\|\frac{\partial^2 U}{\partial t^2}(t')\right\|^2 dt'\right)^{1/2}.\end{aligned}$$

Thus we have from the above estimates that

$$\rho(t) \leqslant C^*(\tau^2 + h^{2(\tilde{r}-1)} + N^{2(1-s)} + \beta).$$

where the constant C^* is described in Theorem 1.

Finally, we show that if h, N^{-1}, τ, and β are sufficiently small and satisfy some conditions, then (6.11) holds for $t_1 = T$. In fact, suppose $\beta = O(\tau^2)$, $h = O(N^{-\mu})$, and $\tau = O(N^{-\lambda})$, with $\lambda, \mu > 0$. If (3.1) holds for $\theta > \sigma/(2\sigma-1)$, then we have

$$\tau N h^{-1}(\tau^2 + h^{2(\tilde{r}-1)} + N^{2(1-s)} + \beta) \to 0, \quad \text{as} \quad h, N^{-1}, \tau \to 0. \tag{6.12}$$

Thus, in this case we have (6.11) for $t_1 = T$. If $\sigma \leqslant \theta/(2\theta-1)$ and (6.8) is satisfied, in addition, then we have (6.12) also. Hence (6.11) holds for $t_1 = T$. Thus we complete the proof of Theorem 1.

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